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Note

A Note on the Rate of Convergence of the Bootstrap

GERHARD DIKTA

Mathematical Institute, University of Giessen, Arndtstrasse 2, D 6300 Giessen, Germany

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It is shown that the bootstrap approximation of the standardized sample mean for the operator introduced by Trotter improves the normal approximation.

1. INTRODUCTION AND MAIN RESULT

Let us assume that $(X_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of random variables, defined on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$, with distribution function (d.f.) F. If the third moment of X_1 exists, $\mathbb{E}(|X_1|^3) < \infty$, the Berry–Esseen theorem guarantees that

$$\sup_{x\in\mathbb{R}} |F_{\eta_n}(x) - F_0(x)| = O(n^{-1/2}), \qquad n\to\infty, \tag{1.1}$$

where F_{η_n} denotes the d.f. of the standardized sample mean $\eta_n = (\sum_{i=1}^n X_i - \mu n)/(\sigma n^{1/2})$, $\mu = \mathbb{E}(X_1)$ the expectation of X_1, $\sigma^2 = \mathrm{Var}(X_1)$ the variance, and F_0 the standard normal d.f.

Under the same assumption, i.e., $\mathbb{E}(|X_1|^3) < \infty$, the proof of Theorem 3 in Butzer, Hahn, and Westphal [1] shows that

$$\sup_{y\in\mathbb{R}} \left| \int f(x+y)\, F_{\eta_n}(dx) - \int f(x+y)\, F_0(dx) \right| = O(n^{-1/2}), \qquad n\to\infty, \tag{1.2}$$

for all $f\in C_b^3 = \{g\in C_b^0 \colon g^{(i)}\in C_b^0,\ 1\leqslant i\leqslant 3\}$, where $g^{(i)}$ denotes the ith derivative of g, and C_b^0 the class of bounded and uniformly continuous functions defined on $\mathbb{R}$.

In his investigation on the asymptotics of Efron's bootstrap Singh [2] proved in Theorem 1.1.5 that the approximation of F_{η_n} by $F_{\eta_n^*}$, the

0021-9045/93 $5.00

corresponding bootstrap d.f., improves the normal approximation. In particular he showed that

$$\sup_{x \in \mathbb{R}} |F_{\eta_n}(x) - F_{\eta_n^*}(x)| = o(n^{-1/2}), \qquad n \to \infty, \tag{1.3}$$

with probability one, if $\mathbb{E}(|X_1|^3) < \infty$, and F is non-lattice. Here, $F_{\eta_n^*}$ denotes the d.f. of the standardized sample mean of the bootstrap sample $(X_i^*)_{i=1,\dots,n}$, which is an i.i.d. sample according to the empirical d.f. F_n of the X-sample.

In this note we prove that the "large O" approximation of (1.2) can also be improved to a "small o" result if the bootstrap d.f. $F_{\eta_n^*}$ is used instead of the normal d.f. F_0.

THEOREM. *Assume that $\mathbb{E}(|X_1|^3) < \infty$. Then, with probability one,*

$$\sup_{y \in \mathbb{R}} \left| \int f(x+y)\, F_{\eta_n}(dx) - \int f(x+y)\, F_{\eta_n^*}(dx) \right| = o(n^{-1/2}), \tag{1.4}$$

$n \to \infty$, for all $f \in C_b^3$.

2. PROOF

Throughout this section we assume w.l.o.g. that $\mathbb{E}(X_1) = 0$. Furthermore, for n fixed we denote by $F_{i,n}$ the d.f. of $X_i/(n^{1/2}\sigma)$ and by $F_{i,n}^*$ the d.f. of $(X_i^* - \mu_n)/(n^{1/2} s_n)$, where $\mu_n = (\sum_{i=1}^n X_i)/n$ and $s_n^2 = (\sum_{i=1}^n X_i^2)/n - \mu_n^2$.

With the same method as that in Section 2 of Butzer, Hahn, and Westphal [1] we get for n fixed by Taylor's expansion of f

$$\begin{aligned}
&\sup_{y \in \mathbb{R}} \left| \int f(x+y)\, F_{\eta_n}(dx) - \int f(x+y)\, F_{\eta_n^*}(dx) \right| \\
&\quad \leqslant n \sup_{y \in \mathbb{R}} \left| \int f(x+y)\, F_{1,n}(dx) - \int f(x+y)\, F_{1,n}^*(dx) \right| \\
&\quad \leqslant n \sup_{y \in \mathbb{R}} |\{\sigma^{-3}\mathbb{E}(X_1^3) - s_n^{-3}\mathbb{E}((X_1^* - \mu_n)^3)\}\, f^{(3)}(y)/(6n^{3/2})| \\
&\qquad + n \sup_{y \in \mathbb{R}} \left| \int \{f^{(3)}(y + \tau_{(x,y)}x) - f^3(y)\}\, x^3/6F_{1,n}(dx) \right| \\
&\qquad + n \sup_{y \in \mathbb{R}} \left| \int \{f^{(3)}(y + \tau_{(x,y)}x) - f^3(y)\}\, x^3/6F_{1,n}^*(dx) \right| \\
&\quad =: n(I_1 + I_2 + I_3),
\end{aligned} \tag{2.1}$$

where $0 \leqslant \tau_{(x,y)} \leqslant 1$.

Now, the SLLN implies that with probability one $I_1 = o(n^{-3/2})$, $n \to \infty$, and as shown in the proof of Theorem 3 in Butzer, Hahn, and Westphal [1], $I_2 = o(n^{-3/2})$, $n \to \infty$. Since $f^{(3)} \in C_b^0$ we can find for an arbitrary $\varepsilon > 0$ a $\delta > 0$ such that $|f^{(3)}(y) - f^{(3)}(z)| < \varepsilon$ if $|y - z| < \delta$. Splitting the integral of I_3 we get

$$I_3 \leqslant \varepsilon \mathbb{E}(|X_1^* - \mu_n|^3)/(6s_n^{3/2}n^{3/2})$$
$$+ 2\|f^{(3)}\| \int |x|^3 1_{\{|x| \geqslant \delta n^{1/2}s_n\}} F_{X_1^* - \mu_n}(dx)/(6s_n^{3/2}n^{3/2}).$$

As in the proof of the Lindeberg condition for Theorem 1.1.1 in Singh [2], $\int |x|^3 1_{\{|x| \geqslant \gamma n^{1/2}s_n\}} F_{X_1^* - \mu_n}(dx) \to 0$, $n \to \infty$, with probability one for all $\gamma > 0$. This together with the SLLN implies that with probability one $I_3 = o(n^{-3/2})$, $n \to \infty$.

References

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